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Decorated Star–Triangle Relations and Exact Integrability of the One-Dimensional Hubbard Model

B. Sriram Shastry[1]

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The exact integrability of the one-dimensional Hubbard model is demonstrated with the help of a novel set of triangle relations, the decorated star–triangle relations. The covering two-dimensional statistical mechanical model obeys the star–triangle or Yang–Baxter relation. A conjecture is presented for the eigenvalues of the transfer matrix.

KEY WORDS: One-dimensional Hubbard model; exactly integrable systems; star–triangle relations.

1. INTRODUCTION

We have recently shown[1] that the one-dimensional (1d) Hubbard model possesses an infinite number of conservation laws by identifying a 2d classical statistical model for which a one-parameter family of transfer matrices commutes with the Hamiltonian. Reference 2 contains a demonstration that the one-parameter family of transfer matrices commute mutually and hence we have a new completely integrable problem. This embedding of the 1d Hubbard model into a covering 2d statistical model parallels the well-known relationship between the 1d XYZ model and the 2d eight-vertex models established by Sutherland and Baxter.[3,4] In this paper we present some further results on the covering statistical model, and also recover the previous results of Refs. 1 and 2 through a promising new line of argument.

The eigenfunctions of the 1d Hubbard model were found by Lieb and Wu[5] using the Bethe–Yang or nested Bethe Ansatz technique. The

[1] Theoretical Physics Section, Tata Institute of Fundamental Research, Bombay-400005, India.

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applicability of the latter suggests the existence of conservation laws. This was emphasized by Heilmann and Lieb,[6] who diagonalized a six-membered, half-filled Hubbard ring and found surprising instances of level crossings and degeneracies.

Barma and Shastry[7] proposed a 2d statistical model with the help of Trotters' formula, for which the 1d Hubbard model is the logarithmic derivative of a transfer matrix [in the sense of Eq. (2.8) to order u]. This model was diagonalized by Bariev[8] through a variant of the coordinate space Bethe ansatz, and elaborated upon by Schotte and Truong.[9,10] This transfer matrix, however, does not commute with the Hamiltonian for general values of Boltzmann weights. In order to find a commuting transfer matrix, one needs further information. In Section 2.1 we present a novel algorithm which yields one nontrivial "current" operator that commutes with the Hamiltonian. This information can be used within the rather tight framework of the transfer matrix formulation (see Luscher[11]) to guess the other members of a commuting family. This is done in Section 2.2, where the form of the transfer matrix is proposed.

Section 3 contains a discussion of a novel class of triangle relations, which we call the decorated star–triangle relations (DSTR). These are intimately related to the star–triangle or Yang–Baxter relations, and rest essentially on the same algebraic structure. The terminology is suggested by the fact that these triangle relations can be pictured as the usual triangle diagrams with additional (diagonal) operators residing on the intermediate lines. However, the DSTRs are an independent set of relations from the STR, and we indicate how one may combine the two in order to get a richer set of STRs. The examples provided in Section 3 yield STRs for the free Fermi vertex models in the presence of fields. These examples are presaged to some extent in the work of Bazhanov and Stroganov,[12] which came to our notice after the completion of this work.

In Section 5, we explore the problem of diagonalizing the transfer matrix. We have not succeeded in an explicit diagonalization, but present a conjecture for the general eigenvalue from which the results of Lieb and Wu follow.

Finally, we mention the review of integrable models by Kulish and Sklyanin,[13] which contains an exhaustive list, and also a discussion of the difficulties of the 1d Hubbard model. Also, the results of Refs. 1 and 2 have been recently verified through a different route by Wadati *et al.*[14,15]

In Section 4 we consider a pair of free Fermi six-vertex models and show that the DSTR together with the STR enable us to construct the R matrix of the covering model for the Hubbard problem rather easily. The R matrix is given explicitly in a compact form.

2. CONSERVED CURRENTS

2.1. Introduction

In this section we present a novel algorithm for identifying conserved currents, i.e., commuting operators with respect to a Hamiltonian expressible in the form

$$H = H_0 + UH_1 \tag{2.1}$$

where H_0 is a "free" Hamiltonian, typically bilinear in fermionic operators, and H_1 is the interaction term quartic in fermionic operators. For the 1d Hubbard model

$$H_0 = -t \sum_{n,\sigma} (C^+_{n+1\sigma} C_{n\sigma} + C^+_{n\sigma} C_{n+1\sigma}) = \sum \varepsilon_{k\sigma} C^+_{k\sigma} C_{k\sigma} \tag{2.2}$$

with

$$C_{k\sigma} = \frac{1}{\sqrt{N}} \sum \exp(ikn)\, C_{n\sigma}, \qquad \varepsilon_k = -2t \cos k$$

and

$$H_1 = \sum n_{m\uparrow} n_{m\downarrow} = \frac{1}{N} \sum C^+_{k+q\uparrow} C_{k\uparrow} C^+_{p-q\downarrow} C_{p\downarrow} \tag{2.3}$$

The summation over m runs from 1 to N, and σ represents the two components of fermions ($\uparrow$ and $\downarrow$). Periodic boundary conditions are assumed everywhere in this work. The XXZ model is in the form (2.1) with U replaced by Δ, and with a single species of fermions.

The free part H_0 commutes with all bilinears in fermions of the form $\sum W_k C^+_{k\sigma} C_{k\sigma}$, and we expect that the currents for H, if they exist, should go over continuously to those of H_0 as $U \to 0$. The simplest Ansatz for a current is

$$j = j_0 + Uj_1 \tag{2.4}$$

where $j_0 = \sum W_k C^+_{k\sigma} C_{k\sigma}$ with some as yet undetermined W_k and j_1. In principle one could go on and add terms to (2.4) of $O(U^2)$, etc. However, we shall truncate at order U, guided by the known results for the XXZ model, where *all* currents are in the above form. Requiring $[j, H] = 0$ and equating various orders of U to zero separately, we find the set of equations

$$[j_0, H_0] = 0 \tag{2.5a}$$

$$[j_0, H_1] = [H_0, j_1^\perp] \tag{2.5b}$$

$$[j_1^\perp, H_1] = -[j_1^\parallel, H_1] \tag{2.5c}$$

Here we have decomposed the operator $j_1 = j_1^{\|} + j_1^{\perp}$, where the parallel part is defined by $[j_1^{\|}, H_0] = 0$.

The algorithm works as follows: we pick some W_k defining j_0 and calculate $j_1^{\perp}$ from (2.5b). This is done most simply by sandwiching (2.5b) between a pair of common eigenstates of H_0 and j_0. The "particle content" of $j_1^{\perp}$ must clearly mimic that of H_1; if H_1 is a four-fermion term, so is $j_1^{\perp}$. The resulting $j_1^{\perp}$ is inserted into the lhs of (2.5c) and one checks if the commutator vanishes or can be simplified to the form of the rhs. It is not guaranteed that (2.5c) can be satisfied; in general, one would have to try various W_k. The working of this scheme is straightforward, if tedious, in momentum space.

We tried this scheme for the Hubbard model and found the following Hermitian currents corresponding to the simplest choices of $W_k = \sin k$ and $\sin 2k$:

$$\begin{aligned} j_A &= (it) \sum (C_{n+1\sigma}^{+} C_{n\sigma} - C_{n\sigma}^{+} C_{n+1\sigma}) \\ &\quad + (iU) \sum_{r=1}^{N} \sum_{n=1}^{N-1} C_{r+n\uparrow}^{+} C_{r+n\downarrow}^{+} C_{r\downarrow} C_{r\uparrow} (-)^{n} \\ &\quad (N = \text{odd integer}) \end{aligned} \tag{2.6}$$

$$\begin{aligned} j_B &= (it) \sum (C_{m+2\sigma}^{+} C_{m\sigma} - C_{m\sigma}^{+} C_{m+2\sigma}) + (iU) \sum (C_{m+1\sigma}^{+} C_{m\sigma} - C_{m\sigma}^{+} C_{m+1\sigma}) \\ &\quad + (iU) \sum_{m\sigma} [C_{m\sigma}^{+}(C_{m+1\sigma} - C_{m-1\sigma}) - (C_{m+1\sigma}^{+} - C_{m-1\sigma}^{+}) C_{m\sigma}]\, n_{m-\sigma} \end{aligned} \tag{2.7}$$

The operator j_B contains a nonvanishing $j_1^{\|}$, whereas j_A does not. The current j_A is tantalizing. First, it makes sense only for N odd. The first term (j_0) has $W_k = \sin k$, the group velocity corresponding to $\varepsilon_k \approx \cos k$, and is in fact the current operator in the usual sense. The second term (j_1) corresponds to a kind of long-ranged backflow of doubly occupied sites. In the sector with no double occupation ($U = \infty$) the commutation of j_A with H was first noticed by Brinkman and Rice,[16] who pointed out that the dc conductivity diverges as a consequence. We expect that j_A should be useful in conductivity calculations for U finite; however, in the remainder of this paper we do not encounter it again. Also, we set $t = 1$ in the following.

2.2. Inferring the Transfer Matrix

In this section we outline the considerations used to guess a transfer matrix embedding of the Hamiltonian. The standard models of 2d classical statistical mechanics, such as the six-vertex model, have a rich algebraic structure, as is well known. The transfer matrix depends on a spectral parameter u, and an expansion in powers of u about a suitable value (say zero) generates an infinite number of conserved currents commuting with the Hamiltonian. Generically we write

$$T(u) = T(0)\left[1 + uH + \frac{u^2}{2!}H^2 + \frac{u^2}{2!}(-i)j + O(u^3)\right] \tag{2.8}$$

where $T(0)$ is the right shift operator. Here H is the Hamiltonian and j the first nontrivial current.

In the case of the 1d Hubbard model the Hamiltonian can be written in the form

$$H = \sum H_{n+1,n} \tag{2.9}$$

$$H_{n+1,n} = (\sigma_n^+ \sigma_{n+1}^- + \sigma_{n+1}^+ \sigma_n^-) + (\tau_n^+ \tau_{n+1}^- + \tau_{n+1}^+ \tau_n^-) + \tfrac{1}{4} U \sigma_n^z \tau_n^z \tag{2.10}$$

(the nonsymmetric definition is convenient in later usage). This form is obtained from (2.1)–(2.3) by subtracting a constant from the original Hamiltonian, corresponding to writing $H_1 = \sum (n_{m\uparrow} - 1/2)(n_{m\downarrow} - 1/2)$, and using a Jordan–Wigner transformation

$$C_{m\uparrow} = (\sigma_1^z \cdots \cdot \sigma_{m-1}^z)\sigma_m^- \tag{2.11a}$$

$$C_{m\downarrow} = (\sigma_1^z \cdots \sigma_N^z)(\tau_1^z \cdots \tau_{m-1}^z)\tau_m^- \tag{2.11b}$$

to eliminate the fermions in favor of two species of Pauli matrices σ and τ. The noninteracting problem $U = 0$ corresponds to a pair of uncoupled XY models. In this case we know that the (free Fermi) six-vertex model transfer matrix commutes with the Hamiltonian for a single species, and hence we expect that the relevant statistical model for the Hubbard problem should consist of two copies of the six-vertex model coupled appropriately. The precise nature of the coupling is the subject of investigation in this section. We will find that an explicit knowledge of j is of great help in this regard.

The transfer matrix is written in the standard form

$$T(u) = \operatorname*{tr}_g [L_{N,g}(u)\, L_{N-1,g}(u) \cdots L_{1,g}(u)] \tag{2.12}$$

where g is the auxiliary space variable, corresponding to the horizontal arrows in the row-to-row transfer matrix. The local scattering matrix $L_{n,g}(u)$ is as yet unspecified, apart from the requirement that when $U = 0$, it must reduce to

$$L_{n,g}(u) \xrightarrow[U \to 0]{} l_{ng}^{(\sigma)}(u) \otimes l_{ng}^{(\tau)}(u) \equiv l_{n,g}(u) \tag{2.13}$$

with

$$l_{n,g}^{(\sigma)}(u)=\frac{a+b}{2}+\frac{a-b}{2}\sigma_n^z\sigma_g^z+c(\sigma_n^+\sigma_g^-+\sigma_g^+\sigma_n^-) \tag{2.14}$$

The weights a, b, and c are parametrized by u and obey the free Fermi condition $a^2+b^2=c^2$. In order to fix ideas, set $a=1$, $b=\sinh(u)$, $c=\cosh(u)$.

Consider an expansion of $T(u)$ through second order in u. First we assume an expansion for $L_{n,g}$,

$$L_{n,g}(u)=P_{n,g}(1+uH_{n,g}+\tfrac{1}{2}u^2B_{n,g}+O(u^3)) \tag{2.15}$$

where $P_{n,g}$ is the permutation operator $[L_{n,g}(u=0)]$. The coefficients of expansion $H_{n,g}$ and $B_{n,g}$ determine the expansion of T as

$$T(u)=T(0)\left[1+u\sum_n H_{n+1,n}+\frac{u^2}{2}\sum_n B_{n+1,n}\right.$$
$$\left.+u^2\sum_{n\geqslant m+1} H_{n,n-1}H_{m,m-1}+O(u^3)\right] \tag{2.16}$$

Using (2.9), we rewrite (2.16) as

$$T^{-1}(0)\,T(u)=1+uH+\frac{u^2}{2}H^2$$
$$+\frac{u^2}{2}\left\{\sum_m (B_{m,m-1}-H^2_{m,m-1})+\sum_m [H_{m,m-1},H_{m-1,m-2}]\right\}+O(u^3) \tag{2.17}$$

This is in the form of (2.8) with the explicit representation

$$(-i)j=\sum_m (B_{m,m-1}-H^2_{m,m-1})+\sum_m [H_{m,m-1},H_{m-1,m-2}] \tag{2.18}$$

To summarize the working so far, we see that if we demand that a transfer matrix exist such that its first two coefficients in an expansion in u give the Hamiltonian of the Hubbard model and a nontrivial current operator commuting with H, then the first two coefficients of expansion of the local $L_{n,g}$ operators are constrained by Eqs. (2.10), (2.15), (2.17), and (2.18). The current j used in (2.18) must be calculated separately (as we did in Section 2.1) and should go over in the $U=0$ limit to the first term of (2.7) [i.e., $W_k\propto\sin(2k)$], since the rhs of (2.18) does so by actual calculation.

Thus, we expect that the current j_B in (2.7) and the form of H should constrain the form of $L_{n,g}$ sufficiently to enable us to guess it. Toward this end, we write (2.7) in terms of the Pauli matrices

$$(-i) j_B = \sum_m [(\sigma^+_{m-1} \sigma^z_m \sigma^-_{m+1} - \text{h.c.} + (\sigma \leftrightarrow \tau)]$$
$$+ \frac{U}{2} \sum_m \{\tau^z_m [(\sigma^+_{m+1} - \sigma^+_{m-1}) \sigma^-_m - (\sigma^-_{m+1} - \sigma^-_{m-1}) \sigma^+_m] + (\sigma \leftrightarrow \tau)\} \tag{2.19}$$

Since we know $H_{n+1,n}$ [Eq. (2.10)] and j_B, we can find [redacted] through (2.18). By straightforward calculation we find

$$H^2_{m+1,m} = \tfrac{1}{2}(1 - \sigma^z_m \sigma^z_{m+1}) + (\sigma \leftrightarrow \tau) + 2(\sigma^+_{m+1} \sigma^-_m + \sigma^+_m \sigma^-_{m+1})$$
$$\times (\tau^+_{m+1} \tau^-_m + \tau^+_m \tau^-_{m+1}) + \tfrac{1}{16} U^2 \tag{2.20}$$

$$[H_{m+1,m}, H_{m,m-1}] = [\sigma^z_m (\sigma^+_{m-1} \sigma^-_{m+1} - \text{h.c.}) + \tfrac{1}{2} U \sigma^z_m \tau^z_m$$
$$\times (\sigma^+_m \sigma^-_{m-1} + \sigma^+_{m-1} \sigma^-_m)] + (\sigma \leftrightarrow \tau) \tag{2.21}$$

From (2.18)–(2.21) we find

$$B_{m,g} = \tfrac{1}{2}(1 - \sigma^z_m \sigma^z_g) + \tfrac{1}{2}(1 - \tau^z_m \tau^z_g) + 2(\sigma^+_m \sigma^-_g + \text{h.c.})(\tau^+_m \tau^-_g + \text{h.c.})$$
$$+ \tfrac{1}{2} U (\tau^+_m \tau^-_g + \tau^+_g \tau^-_m) \sigma^z_g \tau^z_g + \tfrac{1}{2} U (\sigma^+_m \sigma^-_g + \sigma^+_g \sigma^-_m) \sigma^z_g \tau^z_g + \tfrac{1}{16} U^2 \tag{2.22}$$

[we have equated the summands in (2.18) and replaced $m-1$ by g]. The form of the L operator is easy to guess at this stage from Eqs. (2.15), (2.10), and (2.22). In particular, (2.22) indicates that $L_{n,g}$ is probably $l_{n,g}$ postmultiplied by a function with the first derivative equal to $U/4 \sigma^z_g \tau^z_g$ and the second derivative $U^2/16$. We therefore guess

$$L_{ng}(u) = l_{ng}(u) \exp(h \sigma^z_g \tau^z_g) \tag{2.23}$$

where $h = h(u)$, with $h(0) = 0$, $h'(0) U/4$, $h''(0) = 0$.

In Refs. 1 and 2 we showed that a transfer matrix (2.12) with L_{ng} chosen as in (2.11) indeed provides a covering model of the Hubbard model with a proper choice of h and u. In the following sections we provide an alternative and rather compact demonstration of the same results.

3. DECORATED STAR–TRIANGLE RELATIONS AND A FUSION PRINCIPLE

In this section we point out the existence of a modified triangle relation satisfied by the generic eight-vertex model, in addition to the usual

822/50/1-2-5

star–triangle, or Yang–Baxter, relation. The "decorated" STR is an independent relation, which is in some sense a consequence of the structure of the STR, and can be used in conjunction with the latter through a kind of fusion principle to generate new models obeying the STR. As a prelude let us summarize the STR.[4] The relation is encountered when we consider the commutation of two transfer matrices in the form of Eq. (2.12) with different Boltzmann weights (vectors) W_1 and W_2,

$$TT' = \operatorname*{tr}_{g_1 g_2} \overleftarrow{\prod_n} \{L_{ng_1}(W_1)\, L_{ng_2}(W_2)\} \tag{3.1a}$$

$$T'T = \operatorname*{tr}_{g_1 g_2} \overleftarrow{\prod_n} \{L_{ng_1}(W_2)\, L_{ng_2}(W_1)\} \tag{3.1b}$$

The symbol $\prod_n^{\leftarrow}$ stands for an ordered product as in (2.12). The commutator $[T, T']$ vanishes, as first noted by Baxter, when an invertible operator R exists such that

$$L_{32}(W_1)\, L_{31}(W_2)\, R_{12}(W_3) = R_{12}(W_3)\, L_{32}(W_2)\, L_{31}(W_1) \tag{3.2}$$

(writing $n \to 3$, $g_1 \to 2$, $g_2 \to 1$). Writing $R_{12} = P_{12} S_{12}$, with P as the permutation operator, we find

$$L_{31}(W_1)\, L_{32}(W_2)\, S_{12}(W_3) = S_{12}(W_3)\, L_{32}(W_2)\, L_{31}(W_1) \tag{3.3}$$

The form of the operator S_{12} need not in general be the same as that of L; in fact, S may act upon a different kind of Hilbert space[13] from L. The parameters W_3 in general depend on W_1 and W_2 independently and may be indicated in the form $(W_2 \mid W_1)$. Considering the product $L_{0,1}(W_1)\, L_{0,2}(W_2)\, L_{0,3}(W_3)$, there are two distinct ways of rewriting this using (3.3) (corresponding to the two usual "braids"), which implies

$$[L_{03} L_{02} L_{01},\, S_{12}^{-1} S_{13}^{-1} S_{23}^{-1} S_{12} S_{13} S_{23}] = 0$$

Hence we expect

$$\begin{aligned} &S_{31}(W_1 \mid W_3)\, S_{32}(W_2 \mid W_3)\, S_{12}(W_2 \mid W_1) \\ &\quad = S_{12}(W_2 \mid W_1)\, S_{32}(W_2 \mid W_3)\, S_{31}(W_1 \mid W_3) \end{aligned} \tag{3.4}$$

This relation is not strictly a consequence of (3.3), but follows if the product $L_{01} L_{02} L_{03}$ is sufficiently nontrivial.[13] In any case it has to be checked independently.

In the case of the eight-vertex model, the famous result of Baxter[4] is in the form of (3.3) with

$$l_{31}^{8\mathrm{V}}(W_1) = \frac{a_1 + b_1}{2} + \frac{a_1 - b_1}{2} \sigma_1^z \sigma_3^z + \frac{c_1 + d_1}{2} \sigma_1^x \sigma_3^x + \frac{c_1 - d_1}{2} \sigma_1^y \sigma_3^y \tag{3.5}$$

where the Boltzmann weight vector $W_1 = (a_1, b_1, c_1, d_1)$, $l_{32}(W_2)$ is the same as above with $\sigma_1^\alpha \to \sigma_2^\alpha$, and $(a_1 b_1 c_1 d_1) \to (a_2 b_2 c_2 d_2)$. The S_{12} is also in the same form with $\sigma_3^\alpha \to \sigma_2^\alpha$ and $(a_1 \cdots) \to (a_3 \cdots)$. The consistency conditions for the Boltzmann weights are summarized in terms of the invariants

$$\Delta_n \equiv (a_n^2 + b_n^2 - c_n^2 - d_n^2)/2a_n b_n \tag{3.6a}$$

$$\Gamma_n = c_n d_n / a_n b_n \tag{3.6b}$$

The consistency condition becomes

$$\Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta, \qquad \Gamma_1 = \Gamma_2 = \Gamma_3 \equiv \Gamma \tag{3.7}$$

The weights a_3, b_3, c_3, d_3 can be computed explicitly in terms of $(a_2 \cdots)$ and $(a_1 \cdots)$ and are given by

$$\begin{aligned} a_3 &= a_1(c_1 c_2 - d_1 d_2)(b_1^2 c_2^2 - c_1^2 a_2^2)/c_1 \\ b_3 &= b_1(d_1 c_2 - c_1 d_2)(a_1^2 c_2^2 - d_1^2 a_2^2)/d_1 \\ c_3 &= c_1(b_1 b_2 - a_1 a_2)(a_1^2 c_2^2 - d_1^2 a_2^2)/a_1 \\ d_3 &= d_1(a_1 b_2 - b_1 a_2)(b_1^2 c_2^2 - c_1^2 a_2^2)/b_1 \end{aligned} \tag{3.8}$$

(The notation used here differs from that of Baxter[4] in that we use a_1, a_2, a_3, etc., to denote a, a', a'', etc.)

The decorated STRs are given by the relation

$$l_{31}(W_1)\, l_{32}(W_2)\, \sigma_2^z l_{12}(W_4) = l_{12}(W_4)\, \sigma_2^z l_{32}(W_2)\, l_{31}(W_1) \tag{3.9}$$

This is in the form of (3.3) with σ^z inserted in the places indicated. We can easily find the conditions on the Boltzmann weights necessary for (3.9) to hold by noting the identity

$$\begin{aligned} l_{31}(a_1, -b_1, c_1, -d_1) &= \sigma_3^z l_{31}(a_1, b_1, c_1, d_1) \sigma_1^z \\ &= \sigma_1^z l_{31}(a_1, b_1, c_1, d_1) \sigma_3^z \end{aligned} \tag{3.10}$$

Equation (3.10) follows from the definition (3.5) and the usual commutation relations of the Pauli matrices. The four weights are explicitly displayed in (3.10). We use the abbreviation $\bar{W}_n = (a_n, -b_n, c_n, -d_n)$ in the following. Multiplying (3.9) from the left by $\sigma_2^z \sigma_1^z$ and from the right by σ_3^z and using $[l_{32}, \sigma_2^z \sigma_3^z] = 0$, we find

$$l_{31}(\bar{W}_1)\, l_{32}(W_2)\, l_{12}(W_4) = l_{12}(W_4)\, l_{32}(W_2)\, l_{31}(\bar{W}_1) \tag{3.11}$$

This is just the STR with weights $\bar{W}$ in place of W in (3.3). We can thus borrow completely from the previously stated results for STR of Baxter and conclude that the decorated STR (3.9) holds if

$$-\Delta_1 = \Delta_2 = \Delta_4, \qquad \Gamma_1 = \Gamma_2 = \Gamma_4 \tag{3.12}$$

[the invariant Δ_1 changes sign from (3.6a), whereas Γ_1 does not]. The weights a_4, b_4, c_4, and d_4 can be found from (3.8) by merely negating b_1 and d_1 in the rhs.

Thus, the decorated STR connects models with Δ's negated as in (3.2) and does not appear to be very useful in the general case. For the free Fermi case, however, one has independently two sets of triangle relations, Eqs. (3.3) and (3.9), for the *same set* of scattering operators. This fact can be used to advantage, as we now demonstrate in two examples.

3.1. Free Fermi Eight-Vertex Model in a Horizontal Field

Consider the free Fermi case $\Delta_n = 0$, in which case we have both triangle relations (3.3) and (3.9) obeyed. We can add the two with (real) arbitrary coefficients and find the general relation

$$l_{31}(W_1)\, l_{32}(W_2)\, g_{12}^{+} = g_{12} l_{32}(W_2)\, l_{31}(W_1) \tag{3.13}$$

where

$$g_{12} \equiv \alpha l_{12}(W_3) + \beta l_{12}(W_4) \sigma_2^z \tag{3.14}$$

The Hermitian conjugate on the lhs of (3.13) is given by

$$g_{12}^{+} \equiv \alpha l_{12}(W_3) + \beta \sigma_2^z l_{12}(W_4) \tag{3.15}$$

We now observe that (3.13) could be used in the following decorated eight-vertex model, where $L_{31}(W_1) = l_{31}(W_1) I_1$ and $L_{32}(W_2) = l_{32}(W_2) I_2$, with I_1 and I_2 as "decoration" operators acting nontrivially only on the sites 1 and 2. We seek the S operator in (3.3) corresponding to the above L's. Since the (as yet unspecified) operators I_1 and I_2 can be pulled through operators independent of sites 1 and 2, Eq. (3.3) simplifies to

$$l_{31}(W_1)\, l_{32}(W_2)\, I_1 I_2 S_{12} (I_1 I_2)^{-1} = S_{12} l_{32}(W_2)\, l_{31}(W_1) \tag{3.16}$$

Comparing with (3.13)–(3.15), we infer

$$S_{12} = g_{12} \tag{3.17}$$

$$I_1 I_2 g_{12} = g_{12}^{+} I_1 I_2 \tag{3.18}$$

Therefore, if we can find decoration operators I_1 and I_2 and a pair α, β such that Eq. (3.18) is satisfied, then g_{12} is the S operator in the sense of the STR (3.3), with the L operators given by

$$L_{ng}(W_g) = l_{ng}(W_g) I_g \tag{3.19}$$

An inspection of Eq. (3.18) in fact suggests the form of the decoration operators

$$I_1 = \exp(h_1 \sigma_1^z); \qquad I_2 = \exp(h_2 \sigma_2^z) \tag{3.20}$$

The decorated eight-vertex model thus has nontrivial horizontal electric fields. The explicit solution of (3.18) is rather simple. Considering diagonal matrix elements where the spins 1 and 2 are not flipped, the equation is trivially satisfied. The off-diagonal elements $|\uparrow\downarrow\rangle \to |\downarrow\uparrow\rangle$ and $|\uparrow\uparrow\rangle \to |\downarrow\downarrow\rangle$ respectively yield the constraints

$$(\alpha c_3 - \beta c_4)\, e^{h_2 - h_1} = (\alpha c_3 + \beta c_4) e^{h_1 - h_2}$$

and

$$(\alpha d_3 + \beta d_4) e^{-h_1 - h_2} = (\alpha d_3 - \beta d_4) e^{h_1 + h_2} \tag{3.21}$$

Thus,

$$\frac{\alpha}{\beta} \frac{c_3}{c_4} = \frac{\cosh(h_2 - h_1)}{\sinh(h_2 - h_1)}; \qquad \frac{\alpha}{\beta} \frac{d_3}{d_4} = \frac{\cosh(h_2 + h_1)}{\sinh(h_2 + h_1)} \tag{3.22}$$

It is clear from (3.22) that h_1 and h_2 cannot be arbitrary; we eliminate α/β to find

$$\frac{c_3 d_4}{c_4 d_3} = \frac{\tanh(h_2 + h_1)}{\tanh(h_2 - h_1)}$$

Using (3.8) and analogous equations for a_4, b_4, c_4, and d_4 [obtained by ngating b_1 and d_1 in the rhs of (3.8)], we find after some elementary manipulations the constraint

$$\sinh(2h_1) \frac{a_1^2 - b_1^2}{a_1 b_1} = \sinh(2h_2) \frac{a_2^2 - b_2^2}{a_2 b_2} = g \tag{3.23}$$

where g is some fixed constant. To summarize, we have shown that the free Fermi eight-vertex model in a horizontal field with Boltzmann weights obeying $\Delta = 0$ and (3.23) constraining the parameters provides a one-parameter family of commuting transfer matrices. The R matrix (or S matrix) is given by Eq. (3.14), with

$$\beta/\alpha = c_3/c_4 \tanh(h_2 - h_1) = d_3/d_4 \tanh(h_2 + h_1)$$

In terms of the standard elliptic function parametrization of Baxter, we write

$$a_n : b_n : c_n : d_n = \mathrm{sn}(K - iu_n) : \mathrm{sn}(iu_n) : 1 : k\, \mathrm{sn}(iu_n)\, sn(K - iu_n) \qquad (3.24)$$

with $u_3 = u_2 - u_1$ and $u_4 = u_2 + u_1$. Note that the R matrix is not a function of the spectral parameter difference u_3 alone, but depends also on $u_4 = u_2 + u_1$. This is a common feature to all the models discussed in this paper. We also note that the commutation of the transfer matrix of the eight-vertex model in a field with an appropriate XYZ model Hamiltonian was first discussed by Krinsky[17] and is an infinitesimal statement of the above result (corresponding to b_2 small).

3.2. Free Fermi Six-Vertex Model in Arbitrary Fields

Specializing to the six-vertex case $d_1 = d_2 = 0$, the elliptic parametrization degenerates into a trigonometric parametrization and we set

$$a_n = \cos(\theta_n); \qquad b_n = \sin(\theta_n); \qquad c_n = 1 \qquad (3.25)$$

with $\theta_3 = \theta_2 - \theta_1$ and $\theta_4 = \theta_4 + \theta_1$. The entire argument leading to (3.20) is then common, and the only nontrivial matrix element of (3.18) is $|\uparrow\downarrow\rangle \to |\downarrow\uparrow\rangle$, leading to the constraint

$$\beta/\alpha = \tanh(h_2 - h_1) \qquad (3.26)$$

Thus, the horizontal fields h_1 and h_2 are completely arbitrary and the resulting S matrix is (with ρ arbitrary)

$$\begin{aligned} S_{12}(\theta_2 \mid \theta_1) = \rho[&\cosh(h_2 - h_1)\, l_{12}(\theta_2 - \theta_1) \\ &+ \sinh(h_2 - h_1)\, l_{12}(\theta_2 + \theta_1)\, \sigma_2^z] \end{aligned} \qquad (3.27)$$

We note that vertical fields are easily included, since $[l_{12}, \sigma_1^z + \sigma_2^z] = 0$. To see this, write the expected relation

$$\begin{aligned} &L_{31}(\theta_1)[\exp(b_1 \sigma_3^z)]\, L_{32}(\theta_2)[\exp(b_2 \sigma_3^z)]\, \hat{S}_{12} \\ &\quad = \hat{S}_{12} L_{32}(\theta_2)[\exp(b_2 \sigma_3^z)]\, L_{31}(\theta_1) \exp(b_1 \sigma_3^z) \end{aligned} \qquad (3.28)$$

where $L_{31} = l_{31} \exp(h_1 \sigma_1^z)$, etc., and $\hat{S}$ is yet to be calculated [for $b_1 = b_2 = 0$, $\hat{S}$ is given by (3.27)]. Rewrite $b_2 \sigma_3^z = b_2(\sigma_3^z + \sigma_1^z) - b_2 \sigma_1^z$ in the rhs and $b_1 \sigma_3^z = b_1(\sigma_3^z + \sigma_2^z) - b_1 \sigma_2^z$ on the lhs; commuting factors through, we find a common factor $\exp(b_1 + b_2)\sigma_3^z$ on the extreme right of both sides. Canceling and rearranging, we find

$$\begin{aligned} &L_{31} L_{32}[\exp(b_1 \sigma_2^z)]\, \hat{S}_{12} \exp(-b_2 \sigma_1^z) \\ &\quad = [\exp(b_1 \sigma_2^z)]\, \hat{S}_{12}[\exp(-b_2 \sigma_1^z)]\, L_{32} L_{31} \end{aligned} \tag{3.29}$$

Thus, we can choose

$$\hat{S}_{12} = [\exp(-b_1 \sigma_2^z)]\, S_{12} \exp(b_2 \sigma_1^z)$$

In the two examples that we have given here, the L operator is not Hermitian, since the decoration factor appears on one side of l only. However, we can trivially symmetrize using a "gauge transformation" $L' = QLQ^{-1}$ with an appropriate operator Q. Note that the inclusion of vertical fields has made no use of the free Fermi nature of l's.

Finally, in the case of the six-vertex model, I have checked that Eq. (3.4) is also valid in the form

$$\begin{aligned} &S_{31}(\theta_1 \mid \theta_3)\, S_{32}(\theta_2 \mid \theta_3)\, S_{12}(\theta_2 \mid \theta_1) \\ &\quad = S_{12}(\theta_2 \mid \theta_1)\, S_{32}(\theta_2 \mid \theta_1)\, S_{31}(\theta_1 \mid \theta^3) \end{aligned} \tag{3.30}$$

by a brute force calculation. This implies that a more general inhomogeneous model is integrable with a transfer matrix

$$T(\theta \mid \{\theta_n\}) = \operatorname*{tr}_g \overleftarrow{\prod_n} S_{ng}(\theta \mid \theta_n) \tag{3.31}$$

such that $[T(\theta \mid \{\theta_n\}), T(\theta' \mid \{\theta_n\})] = 0$, with the R matrix again given by (3.27). The parameters $\{\theta_n\}$ are arbitrary constants.

4. INTEGRABILITY AND *R* MATRIX FOR THE COVERING MODEL FOR THE ONE-DIMENSIONAL HUBBARD MODEL

In Section 2 we introduced a model of a pair of six-vertex models coupled in a special way. The transfer matrix (2.12) is built out of local scattering operators, which were guessed to be in the form of Eq. (2.17),

$$L_{ng}(\theta) = l_{ng}(\theta) \exp(h \sigma_g^z \tau_g^z) \tag{4.1}$$

with

$$l_{ng}(\theta) = l_{ng}^{(\sigma)}(\theta) \otimes l_{ng}^{(\tau)}(\theta) \tag{4.2}$$

The operators $l^{(\sigma)}$ and $l^{(\tau)}$ are the usual free Fermi six-vertex scattering operators [Eq. (2.14)] parametrized by $a=\cos\theta$, $b=\sin\theta$, $c=1$. The constant h determines the "decoration" operator, i.e., the second factor in (4.1). We now show that the STR can be found for this model in a natural fashion, using the idea of fusion of decorated STRs explained in Section 3.

Let us write down the complete set of STRs and decorated STRs obeyed by the operators $l^{(\sigma)}$ and $l^{(\tau)}$:

$$\begin{aligned} &l_{31}^{(\sigma)}(\theta_1)\, l_{32}^{(\sigma)}(\theta_2)\, l_{12}^{(\sigma)}(\theta_2-\theta_1) \\ &\qquad = l_{12}^{(\sigma)}(\theta_2-\theta_1)\, l_{32}^{(\sigma)}(\theta_2)\, l_{31}^{(\sigma)}(\theta_1) \end{aligned} \tag{4.3}$$

$$\begin{aligned} &l_{31}^{(\sigma)}(\theta_1)\, l_{32}^{(\sigma)}(\theta_2)\, \sigma_2^z l_{12}^{(\sigma)}(\theta_2+\theta_1) \\ &\qquad = l_{12}^{(\sigma)}(\theta_2+\theta_1)\, \sigma_2^z l_{32}^{(\sigma)}(\theta_2)\, l_{31}^{(\sigma)}(\theta_1) \end{aligned} \tag{4.4}$$

We have two more equations of the same form as (4.3) and (4.4) with τ replacing σ. Taking direct products as in (4.2), we write down two resulting equations

$$\begin{aligned} &l_{31}(\theta_1)\, l_{32}(\theta_2)\, l_{12}(\theta_2-\theta_1) \\ &\qquad = l_{12}(\theta_2-\theta_1)\, l_{32}(\theta_2)\, l_{31}(\theta_1) \end{aligned} \tag{4.5}$$

$$\begin{aligned} &l_{31}(\theta_1)\, l_{32}(\theta_2)\, \sigma_2^z\tau_2^z l_{12}(\theta_2+\theta_1) \\ &\qquad = l_{12}(\theta_2+\theta_1)\, \sigma_2^z\tau_2^z l_{32}(\theta_2)\, l_1(\theta_1) \end{aligned} \tag{4.6}$$

Taking a linear combination, we find

$$l_{31}(\theta_1)\, l_{32}(\theta_2)\, g_{12}^{+} = g_{12} l_{32}(\theta_2)\, l_{31}(\theta_1) \tag{4.7}$$

with

$$g_{12} = \alpha l_{12}(\theta_2-\theta_1) + \beta l_{12}(\theta_2+\theta_1)\sigma_2^z\tau_2^z \tag{4.8}$$

[compare with Eqs. (3.13) and (3.14)]. We can now couple the σ and τ models through a "decoration" coupling given in (4.1). Thus, $L_{31} = l_{31}\exp(h_1\sigma_1^z\tau_1^z)$ and $L_{32} = l_{32}\exp(h_2\sigma_2^z\tau_2^z)$, and the expected STR (3.3) gives

$$\begin{aligned} &l_{31}(\theta_1)\, l_{32}(\theta_2)[\exp(h_1\sigma_1^z\tau_1^z + h_2\sigma_2^z\tau_2^z)]\, S_{12} \\ &\qquad = S_{12} l_{32}(\theta_2)\, l_{31}(\theta_1)\exp(h_1\sigma_1^z\tau_1^z + h_2\sigma_2^z\tau_2^z) \end{aligned} \tag{4.9}$$

Comparing with (4.7), we find

$$S_{12} = g_{12} \tag{4.10}$$

$$\exp[(h_1\sigma_1^z\tau_1^z + h_2\sigma_2^z\tau_2^z)]\, g_{12} = g_{12}^{+}\exp(h_1\sigma_1^z\tau_1^z + h_2\sigma_2^z\tau_2^z) \tag{4.11}$$

The condition (4.11) is trivially satisfied for two classes of terms: (1) diagonal in σ and in τ, (2) off-diagonal in both σ and in τ. Nontrivial constraints only arise when we consider a spin flip in one species and a diagonal term in the other. Using the equivalence of the two species, we need to consider two kinds of nontrivial terms $|\{\uparrow\downarrow\}\uparrow\uparrow\rangle \to |\{\downarrow\uparrow\}\uparrow\uparrow\rangle$ (i.e., $c\otimes a$) and $|\{\uparrow\downarrow\}\uparrow\downarrow\rangle \to |\{\downarrow\uparrow\}\uparrow\downarrow\rangle$ (i.e., $c\otimes b$), leading to the equations

$$\frac{\beta}{\alpha}\frac{a_4}{a_3} = \tanh(h_2 - h_1); \qquad \frac{\beta}{\alpha}\frac{b_4}{b_3} = \tanh(h_2 + h_1) \tag{4.12}$$

Eliminating β/α, we find the consistency condition

$$\frac{a_3}{b_3}\tanh(h_2 - h_1) = \frac{a_4}{b_4}\tanh(h_2 + h_1) \tag{4.13}$$

with

$$a_3 = \cos(\theta_2 - \theta_1), \qquad a_4 = \cos(\theta_2 + \theta_1)$$

$$b_3 = \sin(\theta_2 - \theta_1), \qquad b_4 = \sin(\theta_2 + \theta_1)$$

Simplifying further, we find

$$\frac{\sinh(2h_1)}{a_1 b_1} = \frac{\sinh(2h_2)}{a_2 b_2} = \frac{U}{2} \tag{4.14}$$

The constant on the rhs is chosen in conformity with Refs. 1 and 2. The S matrix follows from (4.10) (with ρ arbitrary)

$$\begin{aligned} S_{12}(\theta_2 \mid \theta_1) = {} & = \rho[\cos(\theta_2 + \theta_1)\cosh(h_2 - h_1)\, l_{12}(\theta_2 - \theta_1) \\ & + \cos(\theta_2 - \theta_1)\sinh(h_2 - h_1)\, l_{12}(\theta_2 + \theta_1)\, \sigma_2^z \tau_2^z] \end{aligned} \tag{4.15}$$

In summary, we have seen that the covering model of the 1d Hubbard model, defined by the transfer matrix (2.12) with the L operator given by (4.1), is an integrable system with the coupling h chosen according to (4.14). The S matrix (4.15) is essentially identical to the one found in Ref. 2, and differs only in that we have worked here with a non-Hermitian L operator, a simple "gauge transformation"

$$L' = [\exp(h\sigma_g^z \tau_g^z/2)]\, L \exp(-h\sigma_g^z \tau_g^z/2)$$

recovers completely the previous result.

I have checked some nontrivial matrix elements of (3.30) and believe it to be true in general; I am unable to give a tidy analytical proof of this result. This result implies that an inhomogeneous covering model with a transfer matrix (3.31) is also integrable, with $\{\theta_n\}$ arbitrary.

Some remarks concerning possible generalizations are in order at this stage. We have assumed that the spectral parameters θ_1 and θ_2 are the same for both $l^{(\sigma)}$ and $l^{(\tau)}$. The only other choice permitted is to negate one of the parameters, i.e., to consider $l'(\theta) = l^{(\sigma)}(\theta) \otimes l^{(\tau)}(-\theta)$, for otherwise we obtain too many consistency conditions in the sense of (4.12). The eight-vertex generalization of this scheme fails for the same reason; we end up with too many consistency conditions. One important question has been whether one could couple two general XXZ models in the same sense as the Hubbard model. We see that such a scheme is not possible, since the decorated STRs are useful in the above sense only for the free Fermi case. We have also verified that "natural" generalizations of the Hubbard problem to include more components in a *symmetric* fashion also fail because one obtains too many constraint equations. (The reader is urged to try the three-component problem independently.) Nonsymmetrical couplings might be allowed, althrough I have not checked these in detail.

We note that the model considered is intimately connected, in certain limits, to the isotropic Heisenberg antiferromagnet (XXX model). One correspondence follows from the degeneration to the Hubbard Hamiltonian to first order in θ [Eq. (2.8)], and the relationship of the latter to the XXX model in the limit of U large in the half-filled limit.[18] Another follows in the limit of h large and positive in the sector with all sites having either no particles (i.e., $\downarrow\downarrow$) or two particles ($\uparrow\uparrow$). Here the only allowed arrow configurations are six in number, with identical arrows on the σ and τ lattices, for which the invariant

$$\Delta = (a^4 + b^4 - c^4)/(2a^2b^2) = -1$$

(using $a^2 + b^2 = c^2$).

Let us remark that the STR (3.3) for the covering model has the feature that the infinitesimal limit of $S_{12}(w_3)$ as $w_1 \to w_2$ does not yield H_{12}, thereby sidestepping the difficulty mentioned in Ref. 13 [after Eq. (3.20)].

5. EIGENVALUES OF THE TRANSFER MATRIX

In this section we give a brief and regrettably incomplete account of the eigenvalues of the transfer matrix of the covering model (2.12). The problem is quite nontrivial, from either the coordinate-space Bethe Ansatz point of view or the algebraic Bethe Ansatz point of view.[19–21] One of the main difficulties is the absence of an obvious uniformizing parametrization of the S matrix (4.15), i.e., a parametrization in terms of which *all* the matrix elements are functions of the *difference* of appropriate spectral parameters.

From the algebraic point of view, a central role is ascribed to the global monodromy matrix

$$\Upsilon=\overleftarrow{\prod_n}\{l_{ng}(\theta)\exp(h\sigma_g^z\tau_g^z)\} \tag{5.1}$$

The four-dimensional auxiliary space g may be labeled in the form $(1\rangle=|\uparrow\uparrow\rangle$, $|2\rangle=|\downarrow\downarrow\rangle$, $|3\rangle=|\downarrow\uparrow\rangle$, and $|4\rangle=|\uparrow\downarrow\rangle$, and the matrix elements of Υ are denoted by T_{ij}. The pseudo commutation relations between $T_{ij}(\theta_1)$ and $T_{kl}(\theta_2)$ can be found from the STR (3.2) or (3.3) with the help of the explicit S matrix (4.15). We find a total of 256 relations, which may be written down with considerable labor. The state with all spins up $|\Omega\rangle$ is the vacuum state, and is an eigenfunction of T_{ii} for all i with ($a=\cos\theta$, $b=\sin\theta$)

$$\begin{aligned} T_{11}|\Omega\rangle&=a^{2N}e^{Nh}|\Omega\rangle; \qquad & T_{22}|\Omega\rangle&=b^{2N}e^{Nh}|\Omega\rangle \\ T_{33}|\Omega\rangle&=a^Nb^Ne^{-Nh}|\Omega\rangle; \qquad & T_{44}|\Omega\rangle&=a^Nb^Ne^{-Nh}|\Omega\rangle \end{aligned} \tag{5.2}$$

The difficulty of the problem arises from the proliferation of possible creation operators, T_{21}, T_{31}, T_{41}, T_{23}, T_{24} and composite operators T_{43}, T_{21}, etc. The only (rather trivial) case for which I have been able to construct eigenstates of T explicitly is the one with particles of one species only (say σ species), for which the state $T_{31}(\theta_1)\,T_{31}(\theta_2)\cdots T_{31}(\theta_n)|\Omega\rangle$, or $T_{24}(\theta_1)\,T_{24}(\theta_2)\cdots T_{24}(\theta_n)|\Omega\rangle$, is an exact eigenstate of T. The analysis of the commutation relations is sufficiently tedious and uninspiring as to prevent its inclusion here [the eigenvalues are consistent with Eq. (5.7) given below].

From the coordinate space point of view, the commutation of the transfer matrix with the Hubbard model implies that the eigenstates of the latter are also candidates for the former. Guided by the results of one- and two-spin deviation and the form of the results of the Bethe–Yang Ansatz for the Hubbard model, we conjecture the eigenvalue of the transfer matrix below. We show that the form of the eigenvalue, with the added constraint that poles in the eigenvalue on varying the Boltzmann weights in the finite part of the complex plane have vanishing residues, yields all the subsidiary conditions needed to fix the parameters.

We first parameterize the Boltzmann weights somewhat differently from before; let

$$a=1/(e^{4x}+1)^{1/2}, \qquad b=e^{2x}/(e^{4x}+1)^{1/2}, \qquad c=1 \tag{5.3}$$

whence Eq. (4.14) implies

$$\sinh(2h)\cosh(2x)=U/4 \tag{5.4}$$

Also define a "spin wave" function

$$\sigma_{\pm}(z) \equiv (e^{2x} + ze^{\pm 2h})/(1 - ze^{2x \pm 2h}) \tag{5.5}$$

The functions σ arise in the calculation of the wanted terms in the operation of T on the spin wave states $\sum z^n \sigma_n^- |\Omega\rangle$. In the sector with M particles with $M-K$ particles having spin up and K particles having spin down, the Bethe wave function is written in the form[22]

$$|M, K, \{z_n\}\rangle_Q = \sum_P A(Q \mid P)\, z_{P_1}^{n_1} z_{P_2}^{n_2} \cdots z_{P_M}^{n_M} \rho_{n_1}^{(Q\uparrow)} \cdots \rho_{n_{M-K}}^{(Q\uparrow)} \times \rho_{n_{M-K+1}}^{(Q\downarrow)} \cdots \rho_M^{(Q\downarrow)} |\Omega\rangle \tag{5.6}$$

where z_n are generalized momenta $z_n = e^{ik_n}$, P represents the permutation of the momentum set, and Q is a sector permutation label, $\rho_n^{(\uparrow)} \equiv \sigma_n^-$ and $\rho_n^{(\downarrow)} \equiv \tau_n^-$. For example, the state corresponding to the identity sector $Q = e$ has a string

$$\sigma_{n_1}^- \cdots \sigma_{n_{M-K}}^- \tau_{n_{M-K+1}}^- \cdots \tau_{n_M}^-$$

in (5.6). The amplitudes A are determined in the Hubbard model by requiring that (5.6) be an eigenfunction of H, and the eigenvalue condition determining Z_n requires the nested or Bethe–Yang Ansatz involving a new set of complex numbers $\{\lambda_m\}$, which are K in number.

The eigenvalue of T (actually the adjoint of T) on the state (5.6) is conjectured to be

$$\begin{aligned}
&\Lambda_{M,K}(\theta, \{z_n\}, \{\lambda_m\}) \\
&\quad = a^{2N} e^{Nh} \prod_{n=1}^{M} \sigma_-(z_n) + b^{2N} e^{Nh} (-1)^M \prod_{n=1}^{M} \sigma_+(z_n) \\
&\qquad + a^N b^N e^{-Nh} (-1)^{M-K} \prod_{n=1}^{M} \sigma_-(z_n) \\
&\qquad \times \prod_{m=1}^{K} \left(\frac{e^{2h-2x} - e^{2x-2h} - \lambda_m + U/2}{e^{2h-2x} - e^{2x-2h} - \lambda_m - U/2} \right) + a^N b^N e^{-Nh} (-1)^K \\
&\qquad \times \prod_{n=1}^{M} \sigma_+(z_n) \prod_{m=1}^{K} \left(\frac{e^{-2h-2x} - e^{2x+2h} - \lambda_m - U/2}{e^{-2h-2x} - e^{2x+2h} - \lambda_m + U/2} \right)
\end{aligned} \tag{5.7}$$

Some feeling for the numbers λ can be obtained from the results for $M = 2$, $K = 1$, where $\lambda \propto (z_1 - z_1^{-1} + z_2 - z_2^{-1})$. The conjecture (5.7) can be viewed as a kind of analytical Ansatz in the sense of Reshetekhin.[23] We now list a few important checks, which are fulfilled by (5.7) in support of the conjecture.

1. The limit $U=0$ is trivially satisfied; we merely multiply the eigenvalues of two free Fermi models.

2. The asymptotic behavior of the eigenvalue belonging to a given sector can be readily found for $\theta = -iu$, $u \to$ large and positive. Here

$$a = \cos\theta \to \tfrac{1}{2}e^{u}, \qquad b = \sin\theta \to \tfrac{1}{2}e^{u - i\pi/2}, \qquad c = 1$$

The weights corresponding to the diagonal vertices a and b are dominant and hence spin deviations remain localized. We consider the two cases $U>0$ and $U<0$ separately.

(a) $U>0$. In this case $h>0$ and from $\sinh(2h) = U/2ab$, we find

$$e^{h} \to e^{u - i\pi/4 + \phi/2}, \qquad \text{where} \quad \phi = \ln(U/4)$$

The eigenvalue for $M-K$ spin-up particles and K spin-down particles follows from T_{11} and T_{22}, and is

$$\begin{aligned} \Lambda \to {} & 2^{-2N} e^{3uN + N\phi/2 - iN\pi}(e^{-iM\pi/2} + e^{-i\pi/2(2N-M)}) \\ & + O(e^{(3N-1)u}) \end{aligned} \tag{5.8a}$$

(b) $U<0$. In this case $h<0$ and

$$e^{-h} \to e^{u - i\pi/4 + \phi'/2}, \qquad \phi' = \ln(|U|/4)$$

We find from T_{33} and T_{44}

$$\begin{aligned} \Lambda \to {} & 2^{-2N} e^{3uN + N\phi'/2 - 3i\pi N/4}(e^{-i\pi(M-2K)/2} + e^{-i\pi(2K-M)/2}) \\ & + O(e^{(3N-1)u}) \end{aligned} \tag{5.8b}$$

The eigenvalue (5.7) satisfies (5.8a) and (5.8b), as is readily seen.

3. Consider the adjoint of the transfer matrix T. From Eq. (5.1) the transfer matrix is obtained by taking the trace. We use the cyclic invariance of the trace to insert $1 = (\sigma_g^x \tau_g^x)^2$ and write

$$T(\theta) = \operatorname*{tr}_{g} \sigma_g^x \tau_g^x \Upsilon(\theta)\, \sigma_g^x \tau_g^x$$

Using

$$[\sigma_g^x \tau_g^x, \sigma_g^z \tau_g^z] = 0, \qquad \sigma_g^x \tau_g^x l_{ng}(\theta)\, \sigma_g^x \tau_g^x = l_{ng}^*(\pi/2 - \theta)$$

(where the asterisk denotes Hermitian conjugation in the quantum variables n only), we find

$$T(\theta) = T^+(\pi/2 - \theta) \qquad \text{or} \qquad T^+(\theta) = T(\pi/2 - \theta) \tag{5.9}$$

This incidentally shows that the transfer matrix is normal, i.e., $[T(\theta), T^{+}(\theta')]=0$, since $[T(\theta), T(\theta')]=0$ for all θ, θ'. The eigenvalues therefore must satisfy the condition

$$\bar{\Lambda}(\theta)=\Lambda(\pi/2-\theta) \tag{5.10}$$

where $\bar{\Lambda}$ is the eigenvalue of T^{+}. Equation (5.10) is in fact a constraint on the form of Λ,[23] since $\bar{\Lambda}$ should be alternately deducible from Λ by inverting $z_n \to z_n^{-1}$ (and $\lambda_m \to -\lambda_m$). Thus, we demand

$$\Lambda(\theta, \{z_n\}, \{\lambda_m\})=\Lambda(\pi/2-\theta, \{z_n^{-1}\}, \{-\lambda_m\}) \tag{5.11}$$

Equation (5.7) is readily seen to satisfy (5.11) on using (5.5), from which

$$\sigma_{\pm}(z)\xrightarrow[\theta\to\pi/2-\theta]{z\to z^{-1}}(-1)\,\sigma_{\mp}(z) \tag{5.12}$$

4. We finally note an "inversion relation" satisfied by T. The S matrix (4.15) satisfies the condition

$$S_{12}(\theta_1-\pi/2\,|\,\theta_1)\propto p_{12}^{(-)} \tag{5.13}$$

where $p_{12}^{(-)}$ is the antisymmetrization operator [a direct product of the antisymmetrization operators $p_{12}^{\sigma(-)}\equiv\frac{1}{2}(1-\sigma_1^z\sigma_2^z)-(\sigma_1^+\sigma_2^+ +\text{h.c.})$, and a similar $p_{12}^{\tau(-)}$]. Therefore,

$$\begin{aligned}&L_{ng_1}(\theta_1)\,L_{ng_2}(\theta_1-\pi/2)\,p_{g_1g_2}^{(-)}\\&\quad=p_{g_1g_2}^{(-)}L_{ng_2}(\theta_1-\pi/1)\,L_{ng_1}(\theta_1)\end{aligned} \tag{5.14}$$

Premultiplying by the symmetrization operator $p^{(+)}$, we find

$$p_{g_1g_2}^{(+)}\,L_{ng_1}(\theta_1)\,L_{ng_2}(\theta_1-\pi/2)\,p_{g_1g_2}^{(-)}=0 \tag{5.15}$$

This result has nontrivial consequences for the matrix product

$$T(\theta_1)\;\; T(\theta_1-\pi/2)=\operatorname*{tr}_{g_1g_2}\overleftarrow{\prod_n}\{L_{ng_1}(\theta_1)\,L_{ng_2}(\theta_1-\pi/2)\} \tag{5.16}$$

We observe that the antisymmetric one-dimensional subspace $(\uparrow\downarrow-\downarrow\uparrow)\otimes(\uparrow\downarrow-\downarrow\uparrow)$ corresponding to a product of the singlets in g_1 and g_2 (for the σ and τ species) does not connect to the symmetric subspace and hence a block triangularity results. This argument is similar to that in Ref. 23 for the XXZ model. The matrix element within the 1d subspace is readily computed (using $h_1=-h_2$) and we find

$$T(\theta_1)\,T(\theta_1-\pi/2)=\cos^{4N}(\theta_1)+\tilde{T}(\theta_1) \tag{5.17}$$

The remainder $\tilde{T}$ contains powers of $\sin\theta_1$ and vanishes for $\theta_1 = 0$ [since $T(-\pi/2)$ is the left shift operator, i.e., $T(-\pi/2) = T(0)^{-1}$]. This equation implies a constraint on the eigenvalue Λ, namely the coefficient of a^{4N} in the product $\Lambda(\theta)\,\Lambda(\theta - \pi/2)$ should be unity. This is readily verifid for (5.7) using the result [remembering $h(\theta - \pi/2) = -h(\theta)$]

$$\sigma_{\pm}(z) \xrightarrow{\theta \to \theta - \pi/2} (-1)\frac{1}{\sigma_{\pm}(z)} \tag{5.18}$$

I have verified directly that (5.7) is true for $K = 0$ (by the algebraic Ansatz), but have been unable to prove it in general. Next consider the singularities of Λ arising from fixing Z_n and λ_n and varying a and b, or equivalently x, in the finite part of the complex plane. As stressed by Baxter in his classical paper on the eight-vertex model,[4] such singularities must "go away" somehow, since the eigenfunction of the transfer matrix does not depend on the spectral parameter (θ or x), and hence singularities of the free energy on varying θ must be apparent only. In the case of poles of Λ, one simply demands that the residue should vanish.

The expression (5.7) has poles of two kinds, which we now discuss. The first and third factors have common simple poles corresponding to the vanishing of the denominator of (5.5). Consider one typical term $\sigma_{-}(z_n)$, which blows up for $e^{2x-2h} \to z_n^{-1}$. Equation the residue to zero, setting $(a/be^{2h})^N \to z_n^N$, and canceling common factors, we find the M relations

$$z_n^N = (-1)^{M-K-1} \prod_{m=1}^{K} \left(\frac{z_n - z_n^{-1} - \lambda_m + U/2}{z_n - z_n^{-1} - \lambda_m - U/2}\right) \tag{5.19}$$

The second and fourth terms have common poles from $\sigma_{+}(z_n)$, and equating the residues to zero, one again finds (5.19).

The second class of poles arise from the vanishing denominators of the third and fourth terms. Using (5.4), we see that poles are common, and a typical term has the pole condition $e^{2h-2x} - e^{2x-2h} \to \lambda_n + U/2$. Using the relation

$$\frac{\sigma_{+}(z)}{\sigma_{-}(z)} = \frac{e^{2h-2x} - e^{2x-2h} - (z - z^{-1})}{e^{2h-2x} - e^{2x-2h} - (z - z^{-1}) - U} \tag{5.20}$$

we compute the residue and find the relations

$$\prod_{m \neq n} \left(\frac{\lambda_n - \lambda_m + U}{\lambda_n - \lambda_m U}\right) = (-1)^M \prod_{m} \left(\frac{z_m - z_m^{-1} - \lambda_n - U/2}{z_m - z_m^{-1} - \lambda_n + U/2}\right) \tag{5.21}$$

In order to make contact with the results of Lieb and Wu, we recall

$$\frac{d}{d\theta}\ln T(\theta)\bigg|_{\theta=0} = H = \sum (\sigma_n^+ \sigma_{n+1}^- + \text{h.c.}) + (\sigma \leftrightarrow \tau) + \frac{U}{4}\sum \sigma_n^z \tau_n^z \qquad (5.22)$$

Thus, the largest eigenvalue of T would give the highest energy state of H given above. In order to relate the *lowest energy* states of an appropriate Hamiltonian, we write

$$\bar{H}(|U|) = -\sum (\sigma_n^+ \sigma_{n+1}^- + \text{h.c.}) - \sum (\tau_n^+ \tau_{n+1}^- + \text{h.c.}) + \frac{|U|}{4}\sum \sigma_n^z \tau_n^z \qquad (5.23)$$

We write $U = -|U|$ in (5.22) corresponding to $\sinh(2h) = -\frac{1}{2}|U|ab$ with $a, b > 0$ (thus, $h < 0$ in the principal domain)

$$\frac{d}{d\theta}\ln T(\theta)\bigg|_{\theta=0} = -\bar{H}(|U|) \qquad (5.24)$$

The eigenvalue of $\bar{H}(|U|)$ can be read off from (5.7) easily by noting that as $\theta \to 0$ the first term dominates in the thermodynamic limit. Using $h \to |U|/4\theta + O(\theta^3)$, we readily find the eigenvalue of $\bar{H}$:

$$\bar{e}(M, K, \{z_n\}, \{\lambda_m\}) = (N/4 - M/2)|U| - \sum (z_n + z_n^{-1}) \qquad (5.25)$$

The identification $Z_n \to e^{ik_n}$ recovers the results of Lieb and Wu provided we denote $\lambda_n \to 2i\Lambda_n$ and write $U = -|U|$ in (5.19) and (5.21).

NOTE ADDED IN PROOF

The article "Algebraic Geometry Methods in The Theory of Baxter–Yang Equations" by I. M. Krichever (Mathematical Physics Reviews Vol. 3, ed. S. P. Novikov (Harwood Academic Publisher) 1982), discusses a similar class of S matrices.

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